Ring

Definition → A non Empty Set R, together with two binary Compositions $+$ and $\cdot$ is said to form a Ring if the following axioms are satisfied:

(i) $a+(b+c) = (a+b)+c$ for all $a, b, c \in R$

(ii) $a+b = b+a$ for $a, b \in R$

(iii) $\exists$ some Element 0 (called zero) in R s.t, $a+0 = 0+a = a$ for all $a \in R$

(iv) for Each $a \in R$, $\exists$ an Element $(-a) \in R$ s.t, $a+(-a) = (-a)+a = 0$

(v) $a\cdot(b\cdot c) = a\cdot b + a\cdot c$

(vi) $a\cdot(b+c) = a\cdot b + a\cdot c$

$(b+c)\cdot a = b\cdot a + c\cdot a$ for all $a, b, c \in R$

Remarks :(i) Since we say that $+$ and $\cdot$ are binary compositions on R, it is understood that closure properties w.r.t these hold in R. In other words for all $a, b \in R$, $a+b$ and $a\cdot b$ are unique in R

(ii) One can use any other symbol instead of + and · but for obvious reasons, we use these two symbols (the properties look so natural with these). In fact in future the statement that R is a ring would mean that R has two binary compositions + and · defined on it and satisfies the above axioms.

(iii) Axiom (v) is named associativity w.r.t · and axiom (vi) is referred to as distributivity (left and right) w.r.t + and ·

(iv) Axioms (i) to (iv) could be restated by simply saying that $\langle R, + \rangle$ forms a abelian group.

(v) Since 0 in axiom (iii) is identity w.r.t +, it is clear that this element is unique (Proved in groups)

Q- Check which of the following below is a ring?

(i) Set of real Numbers w.r.t usual addition and Multiplication.

(ii) Set of all Even Integers w.r.t usual addition and Multiplication.

(iii) Set of all 2×2 Matrices over Integers under Matrix addition and Matrix Multiplication.

(iv) Set of all Matrices of the type $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ over Integers under Matrix addition and Multiplication-

(v) the Set $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ under operation addition and Multiplication Modulo 7.