Important

theorem → Every field is an integral domain.

Proof → from definition, we know that field F is a Commutative ring with unity. therefore to prove that every field is an integral domain. we only have to show that field has no zero divisors

i.e for all $a, b \in R$

$$a \cdot b = 0 \Rightarrow a = 0 \text{ or } b = 0$$

Suppose $a \neq 0$, so that a^{-1} exists

then $ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}0$

$$(a^{-1}a)\, b = 0$$

$$(1 \cdot b) = 0$$

$$b = 0$$

Similarly, if $b \neq 0$ and $ab = 0$ then

$$ab = 0 \Rightarrow (ab)\, b^{-1} = 0 \cdot b^{-1}$$

$$\Rightarrow a(bb^{-1}) = 0 \Rightarrow a \cdot 1 = 0$$

$$\Rightarrow a = 0$$

therefore, in a field $ab = 0$

$$\Rightarrow a = 0 \text{ or } b = 0$$

thus, a field has no zero divisors.

Scanned by CamScanner

Hence Every field is an integral domain.
But the Converse of the above theorem is not true.
i.e Every integral domain is not a field.
for Example; the ring of integers is an integral domain but it is not a field. the only inversible Elements of this ring are 1 and -1.

Q→ Give an Example of a Skew field which is not a field?

Solution→ Consider M = Set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are real numbers. the Set M is a Skew field with respect to the addition and Multiplication of Matrices.

Properties Under Addition:

1.) Closure Law: the Sum of two members of M is also a member of M. therefore closure law is satisfied.

2.) Associative Law →

$A + (B + C) = (A + B) + C$ for all $A, B, C \in M$

3.) Existence of identity →

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M$ and is the identity for Matrix addition in M i.e $O + A = A + O = A$ for all $A \in M$

4.) Existence of inverse → for each $A \in M$, there exists a Matrix $-A \in M$ such that

$(-A) + A = O_{2 \times 2}$

5.) Commutative Law → $A + B = B + A$ for all $A, B \in M$

Properties Under Multiplication :

6.) Closure Law : the Product of two Matrices of type 2×2 is a Matrix of type 2×2 i.e $A, B \in M \Rightarrow A \cdot B \in M$

7.) Associative Law → As Matrix Multiplication is Associative

$A(BC) = (AB)C$ for all $A, B, C \in M$

8.) Existence of identity → If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $I \cdot A = A \cdot I = A$ for all $A \in M$

Also $I \in M$ give that I is identity for Matrix Multiplication in M.

9.) Existence of Multiplicative inverse:

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where A is a non Singular Matrix.

$\Rightarrow |A| \neq 0$ i.e A^{-1} Exists and $A^{-1} \in M$.

Also $A \cdot A^{-1} = A^{-1} A = I$

Hence, Each non Singular Matrix possesses Inverse.

10.) Distributive Law: for all Matrices $A, B, C \in M$, we have

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

and $$(B + C) \cdot A = B \cdot A + C \cdot A$$

Hence $(M, +, \cdot)$ is a Skew field. But Matrix Multiplication is not Commutative

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where $A, B \in M$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

$$BA = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix}$$

Hence $AB \neq BA$

So $(M, +, \cdot)$ is a Skew field but not a field.

Scanned by CamScanner